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Graded Differential Polynomial Rings

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Correspondence: Yassine Ait Mohamed (yassine.ait.mohamed@usherbrooke.ca)**Received:** 25 May 2026 | **Revised:** 18 September 2026 | **Accepted:** 7 October 2026**Keywords:** differential polynomial rings | gr-Noetherian rings | gr-primality | gr-simplicity | homogeneous derivations

ABSTRACT

We study differential polynomial rings $R[t; \delta]$ over Γ -graded rings, where Γ is an arbitrary group. We show that $R[t; \delta]$ admits a Γ -grading compatible with that of R if and only if δ is a γ -derivation for some $\gamma \in C_\Gamma(\Gamma_R)$, and that this grading is unique once $\deg(t) = \gamma$ is fixed; if $\delta \neq 0$, then γ is itself uniquely determined by δ . We characterize the resulting graded ring by a universal property. We prove a characteristic-free center criterion for gr-simplicity whenever $Z(R[t; \delta])$ is a graded subring; in characteristic zero, gr-simplicity is equivalent to δ -gr-simplicity of R and γ -outerness of δ , extending Jordan's simplicity criterion to the graded setting. We further show that $R[t; \delta]$ is gr-prime if and only if R is δ -gr-prime, and that gr-Noetherianity of R passes to $R[t; \delta]$, recovering a graded Hilbert basis theorem as a special case. When Γ is abelian, gr-simplicity and gr-primality are shown to be invariants of homogeneous graded Morita equivalence, and every ring homogeneously graded equivalent to $R[t; \delta]$ via a compatible idempotent is again a graded differential polynomial ring.

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1 | Introduction

Throughout the paper, all rings are assumed to be associative with identity, and all ring homomorphisms preserve the identity.

Let R be a ring and δ a derivation of R . The differential polynomial ring $R[t; \delta]$ is the free left R -module with basis $\{1, t, t^2, \dots\}$, with multiplication defined by the rule

$$tx = xt + \delta(x) \quad \text{for all } x \in R, \quad (1.1)$$

extended by associativity and distributivity [1]. Iterating (1.1) yields

$$t^n x = \sum_{k=0}^n \binom{n}{k} \delta^k(x) t^{n-k} \quad (x \in R, n \geq 0), \quad (1.2)$$

where $\delta^0 = \text{id}$. When $\delta = 0$, this reduces to the ordinary polynomial ring $R[t]$.

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Since ordinary polynomial rings are the special case $\delta = 0$, the compatibility problem considered here is related to earlier work on gradings of polynomial rings. In grading frameworks more general than group gradings, including S -graded rings inducing S , compatibility questions for polynomial rings were considered by Chadeyras [2] and Krasner [3]; see also Kelarev [4] for a systematic treatment of graded ring constructions. Ilić-Georgijević [5] extended related constructions to semigroup rings and studied graded radicals of graded polynomial rings. These works provide a natural classical context for the compatibility question studied below.

Differential polynomial rings $R[t; \delta]$ are among the most studied objects in noncommutative algebra [6, 7]. They arise as special cases of Ore extensions and crossed products, and include enveloping algebras of solvable Lie algebras. In the ungraded setting, chain conditions on $R[t; \delta]$ were studied by Cauchon [8], and principality was established by Lam [9]. Simplicity was first characterized by Jordan [10], who showed that when $\text{char}(R) = 0$, the ring $R[t; \delta]$ is simple if and only if R is δ -simple and δ is outer; this criterion was extended to arbitrary characteristic by Öinert and Silvestrov [11]. Primality of $R[t; \delta]$ was studied by Jordan [12], who proved that $R[t; \delta]$ is prime if and only if R is δ -prime. In the graded setting, however, these properties have not been systematically studied for rings graded by an arbitrary group.

This paper addresses two questions. Given a Γ -graded ring $R = \bigoplus_{\tau \in \Gamma_R} R_\tau$, where $\Gamma_R := \{\tau \in \Gamma : R_\tau \neq 0\}$ denotes the support of the grading, and a derivation $\delta \in \text{Der}(R)$, under what conditions does $R[t; \delta]$ admit a Γ -grading compatible with that of R ? And do the classical results on simplicity, primality, and chain conditions extend to this graded setting?

We assume throughout that R is graded by an arbitrary group Γ .

Definition 1.3 (Good grading). A Γ -grading on $R[t; \delta]$ is called a *good grading relative to the given grading of R* if t is homogeneous, say $\deg(t) = \gamma \in \Gamma$, and the following two conditions hold:

- (i) the canonical inclusion $\iota : R \hookrightarrow R[t; \delta]$ is a graded ring homomorphism;
- (ii) the defining relation (1.1) is homogeneous.

Since a general derivation need not preserve homogeneous components, we restrict to homogeneous derivations, that is, derivations δ for which there exists a map $\varphi_\delta : \Gamma_\delta \rightarrow \Gamma$, where

$$\Gamma_\delta := \{\tau \in \Gamma_R : \delta(R_\tau) \neq 0\}$$

denotes the support of δ , satisfying $\delta(R_\tau) \subseteq R_{\varphi_\delta(\tau)}$ for all $\tau \in \Gamma_R$ [13–15]. The following example illustrates the range of gradings that arise and motivates the construction developed in this paper.

The first Weyl algebra $A_1(k) = k\langle x, y \rangle / (yx - xy - 1)$ over a field k carries a natural $\mathbb{Z}$ -grading with $\deg(x) = 1$ and $\deg(y) = -1$ [16], and is isomorphic as a ring to the differential polynomial ring $k[x][y; \frac{d}{dx}]$, where $k[x]$ is graded by $\deg(x) = 1$ [17]. The commutation relation $yr = ry + \frac{d}{dx}(r)$ for $r \in k[x]$ reproduces $yx = xy + 1$. This isomorphism transports the grading of $A_1(k)$ to a $\mathbb{Z}$ -grading on $k[x][y; \frac{d}{dx}]$ in which each monomial $x^a y^i$ has degree $a - i$. In particular, $\deg(x) = 1$, $\deg(y) = -1$, and $\frac{d}{dx}$ shifts degree by -1 , that is, $\frac{d}{dx}(k[x]_n) \subseteq k[x]_{n-1}$. This suggests considering homogeneous derivations satisfying $\delta(R_\tau) \subseteq R_{\tau\gamma}$ for all $\tau \in \Gamma_R$, for a fixed $\gamma \in C_\Gamma(\Gamma_R)$. In this case, setting $\deg(t) = \gamma$ and

$$R[t; \delta]_\tau := \bigoplus_{i \geq 0} R_{\tau\gamma^{-i}} t^i \quad (\tau \in \Gamma) \quad (1.4)$$

yields a good grading on $R[t; \delta]$ (Proposition 2.1). Moreover, in the same proposition we show that any homogeneous derivation for which a good grading on $R[t; \delta]$ exists arises in this form.

We now fix the terminology used in the statements of the main results.

Definition 1.5 (Basic graded-ring terminology). Let Γ be a multiplicatively written group with identity element e . A ring R is a Γ -graded ring if

$$R = \bigoplus_{\sigma \in \Gamma} R_\sigma, \quad R_\sigma R_\tau \subseteq R_{\sigma\tau}, \quad 1_R \in R_e.$$

Its *support* is $\Gamma_R := \{\sigma \in \Gamma : R_\sigma \neq 0\}$, and $\mathcal{H}(R) := \bigcup_{\sigma \in \Gamma} R_\sigma$ denotes the set of homogeneous elements. We write

$$C_\Gamma(\Gamma_R) := \{\gamma \in \Gamma : \gamma\sigma = \sigma\gamma \text{ for every } \sigma \in \Gamma_R\}$$

for the centralizer of the support and $Z(\Gamma)$ for the center of Γ .

An ideal $\mathfrak{a}$ of R is *graded* if

$$\mathfrak{a} = \bigoplus_{\sigma \in \Gamma} (\mathfrak{a} \cap R_\sigma).$$

The ring R is *gr-simple* if it has no nonzero proper graded ideals, *gr-prime* if $\mathfrak{a}\mathfrak{b} = 0$ for graded ideals $\mathfrak{a}, \mathfrak{b}$ implies $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$, and *gr-right-Noetherian* (resp. *gr-left-Noetherian*) if every ascending chain of graded right (resp. left) ideals stabilizes. A ring homomorphism $f : R \rightarrow S$ between Γ -graded rings is *graded* if $f(R_\sigma) \subseteq S_\sigma$ for every $\sigma \in \Gamma$; a bijective graded homomorphism is a *graded isomorphism*, written $R \cong_{\text{gr}} S$. A *graded field* is a graded ring in which every nonzero homogeneous element is invertible. We write $\text{Aut}_{\text{gr}}(R)$ for the group of graded automorphisms of R .

Definition 1.6 (Homogeneous and γ -derivations). A derivation $\delta : R \rightarrow R$ is *homogeneous* if there exists a map

$$\varphi_\delta : \Gamma_\delta \rightarrow \Gamma, \quad \Gamma_\delta := \{\tau \in \Gamma_R : \delta(R_\tau) \neq 0\},$$

such that $\delta(R_\tau) \subseteq R_{\varphi_\delta(\tau)}$ for every $\tau \in \Gamma_R$ [13, 14]. Fix $\gamma \in C_\Gamma(\Gamma_R)$. A homogeneous derivation δ satisfying

$$\delta(R_\tau) \subseteq R_{\tau\gamma} \quad (\tau \in \Gamma_R)$$

is called a γ -*derivation*, and we write $\delta \in \text{Der}_\gamma(R)$. It is γ -*inner* if $\delta = \delta_r$ for some $r \in R_\gamma$, where

$$\delta_r(x) := rx - xr,$$

and γ -*outer* otherwise.

A graded ideal $\mathfrak{a}$ of R is δ -*invariant* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. The ring R is δ -*gr-simple* if its only δ -invariant graded ideals are 0 and R , and δ -*gr-prime* if $\mathfrak{a}\mathfrak{b} = 0$ for δ -invariant graded ideals $\mathfrak{a}, \mathfrak{b}$ implies $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$.

Definition 1.7 (Graded R -rings and the category $\mathcal{C}_\delta$). Let R be a Γ -graded ring. A *graded R -ring* is a pair (S, ϕ) , where S is a Γ -graded ring and

$$\phi : R \rightarrow S$$

is a unital graded ring homomorphism; ϕ is called the *structure map*. If (S, ϕ) and (S', ϕ') are graded R -rings, a morphism of graded R -rings is a graded ring homomorphism $h : S \rightarrow S'$ satisfying $h \circ \phi = \phi'$.

For $\delta \in \text{Der}_\gamma(R)$, let $\mathcal{C}_\delta$ be the category whose objects are triples (S, ϕ, u) consisting of a graded R -ring (S, ϕ) and a homogeneous element $u \in S_\gamma$ such that

$$u\phi(x) - \phi(x)u = \phi(\delta(x)) \quad \text{for every } x \in R. \quad (1.8)$$

A morphism

$$h : (S, \phi, u) \rightarrow (S', \phi', u')$$

in $\mathcal{C}_\delta$ is a morphism of graded R -rings satisfying $h(u) = u'$.

For a subring A of a ring B , we use the notation

$$C_B(A) := \{b \in B : ba = ab \text{ for all } a \in A\}$$

for the centralizer of A in B .

The main results of this paper are as follows.

Theorem 1.9 (Universal property). *Let R be a Γ -graded ring and $\delta \in \text{Der}_\gamma(R)$ with $\gamma \in C_\Gamma(\Gamma_R)$. The pair $(R[t; \delta], t)$ is the initial object of the category C_δ . In particular:*

- (1) *conjugate γ -derivations yield isomorphic graded rings;*
- (2) *an inner shift of δ by an element of R_γ leaves $R[t; \delta]$ unchanged up to graded isomorphism;*
- (3) $M_n(R)[t; \hat{\delta}] \cong_{\text{gr}} M_n(R[t; \delta])$.

Theorem 1.10 (Gr-simplicity). *Let R be a Γ -graded ring, $\gamma \in C_\Gamma(\Gamma_R)$, and $\delta \in \text{Der}_\gamma(R)$.*

- (1) *If R is gr-simple of characteristic zero and δ is γ -outer, then $R[t; \delta]$ is gr-simple.*
- (2) *If $\text{char } R = 0$, then $R[t; \delta]$ is gr-simple if and only if R is δ -gr-simple and δ is γ -outer.*
- (3) *Suppose that $Z(R[t; \delta])$ is a graded subring of $R[t; \delta]$. Then $R[t; \delta]$ is gr-simple if and only if R is δ -gr-simple and $Z(R[t; \delta])$ is a graded field.*
- (4) *Gr-simplicity of $R[t; \delta]$ is preserved under conjugation of δ , inner shifts by elements of R_γ , and matrix extensions.*
- (5) *If R is gr-simple of characteristic zero, then $R[t; 0][s; \frac{d}{dt}]$ is gr-simple.*

Part (2) generalizes Jordan's criterion [10] to the graded setting. Part (3) removes the characteristic zero restriction at the cost of a center condition; it is a graded analogue of [11, Theorem 4.15]. The additional hypothesis that $Z(R[t; \delta])$ be graded is automatic, for example, when Γ is abelian [18, Lemma 2.2], and more generally when Γ admits a total order invariant under left and right multiplication and every nonzero homogeneous element of $R[t; \delta]$ is regular (i.e., neither a left nor a right zero-divisor) [19, Proposition 2.1]. Part (4) shows that gr-simplicity is stable under the structural operations of Theorem 1.9. Part (5) shows that gr-simplicity persists through iterated differential constructions.

Theorem 1.11 (Gr-primality). *Let R be a Γ -graded ring, $\gamma \in C_\Gamma(\Gamma_R)$, and $\delta \in \text{Der}_\gamma(R)$.*

- (1) *If R is gr-prime, then $R[t; \delta]$ is gr-prime.*
- (2) *$R[t; \delta]$ is gr-prime if and only if R is δ -gr-prime.*

This is a graded analogue of [12, Theorem 2.2].

Theorem 1.12 (Gr-Noetherianity). *Let $\delta \in \text{Der}_\gamma(R)$. If R is gr-right-Noetherian (resp. gr-left-Noetherian), then so is $R[t; \delta]$. In particular, taking $\delta = 0$ recovers the graded Hilbert basis theorem.*

In [20], Abrams et al. establish, for rings graded by an *abelian* group, the equivalence of four conditions (HG1)–(HG4) constituting the Homogeneously Graded Extended Morita Theorem [20, Theorem 4.7]. Two of these conditions are purely algebraic:

$$A \cong_{\text{gr}} \ell M_n(B) \ell, \quad (\text{C2})$$

$$M_\infty(A) \cong_{\text{gr}} M_\infty(B), \quad (\text{C4})$$

for some $n \in \mathbb{N}$ and full homogeneous idempotent $\ell \in M_n(B)_e$, where all matrix rings carry the *standard grading* $(M_X(A))_\sigma := M_X(A_\sigma)$. The equivalence (C2) $\Leftrightarrow$ (C4) is proved in [20, Theorem 2.14] through degree- e idempotents and matrix operations assembled via direct limits; the argument never invokes commutativity of Γ . The abelian hypothesis of [20] is therefore a global convention for the categorical parts of their theorem and plays no role in the proof of (C2) $\Leftrightarrow$ (C4). E. Ruiz has indicated that the full Theorem 4.7 of [20], including the categorical conditions, is expected to extend to arbitrary groups (E. Ruiz, private communication, 2026). Since a proof of this extension is not yet available, we work throughout with Γ abelian and recall the following definition.

Definition 1.13. Let Γ be an abelian group and let A, B be Γ -graded rings. We say A and B are *homogeneously graded equivalent*, written $A \approx_{\text{hgr}} B$, if there exist $n \in \mathbb{N}$ and a full homogeneous idempotent $\ell \in M_n(B)_e$ such that $A \cong_{\text{gr}} \ell M_n(B) \ell$, or equivalently $M_\infty(A) \cong_{\text{gr}} M_\infty(B)$ [20, Theorem 2.14].

Under the hypothesis that Γ is abelian, we prove the following results.

Theorem 1.14 (Morita invariance). *Let R be a graded ring and $\delta \in \text{Der}_\gamma(R)$.*

- (1) *If $A \approx_{\text{hgr}} R[t; \delta]$, then A is gr-simple (resp. gr-prime) whenever $R[t; \delta]$ is.*
- (2) *If $A \approx_{\text{hgr}} R[t; \delta]$ arises from a full idempotent $\ell \in R_e$ with $\delta(\ell) = 0$ (so that δ restricts to a γ -derivation on $\ell R \ell$), then $A \cong_{\text{gr}} (\ell R \ell)[t; \bar{\delta}]$, where $\bar{\delta}(s) := \ell \delta(s) \ell$.*

Part (2) shows that the differential polynomial structure is itself an invariant of $\approx_{\text{hgr}}$: Every ring homogeneously graded equivalent to $R[t; \delta]$ via a compatible idempotent is again a graded differential polynomial ring.

The paper is organized as follows. In Section 2, we construct the graded differential polynomial ring $R[t; \delta]$. We characterize the derivations for which a good grading exists (Proposition 2.1), establish the universal property of this construction (Theorem 2.6), and derive the structural isomorphisms given in Corollaries 2.7–2.9. In Section 3, we establish the graded ring-theoretic properties of $R[t; \delta]$: We prove gr-simplicity (Theorems 1.10(1)–(5)), gr-primality (Theorem 1.11; see also Section 3.2), and gr-Noetherianity (Theorem 1.12). In Section 4, we treat the case where the grading group is abelian. We prove that gr-simplicity and gr-primality are invariants of homogeneous graded Morita equivalence (Proposition 4.1), and we show that the differential polynomial ring structure itself is preserved under equivalence via a compatible idempotent (Proposition 4.2, Theorem 1.14).

2 | Construction of Graded Differential Polynomial Rings

We begin by constructing the graded differential polynomial ring $R[t; \delta]$ when the derivation shifts degrees by a fixed element of the centralizer of the support of the grading. We describe the grading on $R[t; \delta]$, explain the universal mapping property that characterizes the construction, and introduce the leading-coefficient ideals that will serve as the main technical tool.

Proposition 2.1. *Let $\gamma \in C_\Gamma(\Gamma_R)$ and define $R[t; \delta]_\tau$ by (1.4) with $\deg(t) = \gamma$.*

- (1) *If δ is a γ -derivation of R , then (1.4) defines a good grading on $R[t; \delta]$.*
- (2) *If δ is a homogeneous derivation such that there exists a good Γ -grading on $R[t; \delta]$, then δ is a γ -derivation.*

Proof. (1) Assume $\gamma \in C_\Gamma(\Gamma_R)$ and $\delta \in \text{Der}_\gamma(R)$. We first show that $R[t; \delta] = \bigoplus_{\tau \in \Gamma} R[t; \delta]_\tau$. Since $R[t; \delta]$ is the free left R -module with basis $\{1, t, t^2, \dots\}$ and $R = \bigoplus_{\sigma \in \Gamma_R} R_\sigma$, every element of $R[t; \delta]$ has a unique expression $\sum_{i \geq 0} r_i t^i$ with $r_i \in R$, and each r_i decomposes uniquely as $r_i = \sum_{\sigma \in \Gamma_R} (r_i)_\sigma$ with $(r_i)_\sigma \in R_\sigma$. Setting $\tau := \sigma \gamma^i$, the term $(r_i)_{\tau \gamma^{-i}} t^i$ belongs to $R_{\tau \gamma^{-i}} t^i \subseteq R[t; \delta]_\tau$, so $R[t; \delta] = \sum_{\tau \in \Gamma} R[t; \delta]_\tau$. To see the sum is direct, suppose $\sum_\tau f_\tau = 0$ with $f_\tau = \sum_{i \geq 0} c_{\tau, i} t^i \in R[t; \delta]_\tau$ and $c_{\tau, i} \in R_{\tau \gamma^{-i}}$. The coefficient of t^i gives $\sum_\tau c_{\tau, i} = 0$ in R . Since distinct values of τ correspond to distinct homogeneous components $R_{\tau \gamma^{-i}}$ of R , the direct-sum decomposition of R forces $c_{\tau, i} = 0$ for all τ and i , hence $f_\tau = 0$ for all τ . Therefore

$$R[t; \delta] = \bigoplus_{\tau \in \Gamma} R[t; \delta]_\tau.$$

It remains to verify conditions (i) and (ii). For (i), if $x \in R_\tau$ then $\iota(x) = x t^0 \in R_{\tau \gamma^0} t^0 = R_\tau t^0 \subseteq R[t; \delta]_\tau$, so ι is graded. For (ii), let $f = x t^i \in R[t; \delta]_\tau$ with $x \in R_{\tau \gamma^{-i}}$ and $g = x' t^j \in R[t; \delta]_\sigma$ with $x' \in R_{\sigma \gamma^{-j}}$. By (1.2),

$$fg = \sum_{k=0}^i \binom{i}{k} x \delta^k(x') t^{i+j-k}.$$

Since $\delta \in \text{Der}_\gamma(R)$, we have $\delta^k(x') \in R_{\sigma \gamma^{k-j}}$, and since $\gamma \in C_\Gamma(\Gamma_R)$,

$$x \delta^k(x') \in R_{\tau \gamma^{-i}} R_{\sigma \gamma^{k-j}} \subseteq R_{\tau \sigma \gamma^{-(i+j-k)}},$$

which is precisely the $(\tau\sigma)$ -component of (1.4) at index $i + j - k$. Hence $fg \in R[t; \delta]_{\tau\sigma}$, so the grading is multiplicative. In particular, for $x \in R_\tau$, both sides of (1.1) are homogeneous of degree $\tau\gamma$: The left-hand side tx lies in $R[t; \delta]_{\gamma\tau} = R[t; \delta]_{\tau\gamma}$ since $\gamma \in C_\Gamma(\Gamma_R)$, the term xt lies in $R[t; \delta]_{\tau\gamma}$, and $\delta(x) \in R_{\tau\gamma} \subseteq R[t; \delta]_{\tau\gamma}$ since $\delta \in \text{Der}_\gamma(R)$. Hence the relation (1.1) is homogeneous of degree $\tau\gamma$ for every $\tau \in \Gamma_R$, and (1.4) is a good grading.

(2) Suppose δ is a homogeneous derivation and that $R[t; \delta]$ admits a good Γ -grading with $\deg(t) = \gamma$. Let $\tau \in \Gamma_R$ and $x \in R_\tau$ be nonzero. By condition (i), $x \in R[t; \delta]_\tau$, and since $t \in R[t; \delta]_\gamma$, the grading axiom gives

$$tx \in R[t; \delta]_{\gamma\tau}, \quad xt \in R[t; \delta]_{\tau\gamma}.$$

By condition (ii), the defining relation is homogeneous of degree $\gamma\tau$, hence tx and $\delta(x)$ must both lie in the homogeneous component containing tx . Since $\tau \in \Gamma_R$ was arbitrary, $\gamma \in C_\Gamma(\Gamma_R)$. Moreover, all three terms have degree $\tau\gamma$, so $\delta(x) \in R[t; \delta]_{\tau\gamma}$. Since $\delta(x) \in R$ and t is graded, $R \cap R[t; \delta]_{\tau\gamma} = R_{\tau\gamma}$, hence $\delta(x) \in R_{\tau\gamma}$. Since $\tau \in \Gamma_R$ and $x \in R_\tau$ were arbitrary, $\delta \in \text{Der}_\gamma(R)$. $\square$

The preceding proposition determines when a good grading exists. Once the degree of t is fixed, this grading is unique.

Proposition 2.2. *Let $\gamma \in C_\Gamma(\Gamma_R)$ and $\delta \in \text{Der}_\gamma(R)$. The grading (1.4) is the unique Γ -grading on $R[t; \delta]$ for which the canonical inclusion $\iota : R \hookrightarrow R[t; \delta]$ is graded and t is homogeneous of degree γ . If $\delta \neq 0$, then γ is uniquely determined by δ .*

Proof. Let

$$R[t; \delta] = \bigoplus_{\tau \in \Gamma} A_\tau$$

be such a grading, and set $B_\tau := \bigoplus_{i \geq 0} R_{\tau\gamma^{-i}} t^i$. If $x_i \in R_{\tau\gamma^{-i}}$, then $x_i \in A_{\tau\gamma^{-i}}$ and $t^i \in A_{\gamma^i}$, hence $x_i t^i \in A_\tau$. Thus $B_\tau \subseteq A_\tau$. Conversely, let $f \in A_\tau$ and write $f = \sum_{i \geq 0} x_i t^i$, with $x_i = \sum_{\sigma \in \Gamma} (x_i)_\sigma t^i$. Each $(x_i)_\sigma t^i$ is homogeneous of degree $\sigma\gamma^i$. Grouping terms of equal degree gives $f = \sum_{\eta \in \Gamma} g_\eta$ with $g_\eta \in A_\eta$. Since $f \in A_\tau$, uniqueness of the homogeneous decomposition gives $g_\eta = 0$ for $\eta \neq \tau$. Therefore

$$f = g_\tau = \sum_{i \geq 0} (x_i)_{\tau\gamma^{-i}} t^i \in B_\tau.$$

Hence $A_\tau = B_\tau$ for every $\tau \in \Gamma$.

Finally, suppose $\delta \neq 0$ and $\delta \in \text{Der}_{\gamma'}(R)$ for some $\gamma' \in C_\Gamma(\Gamma_R)$. Choose $x \in R_\sigma$ with $\delta(x) \neq 0$. Then $\delta(x) \in R_{\sigma\gamma} \cap R_{\sigma\gamma'}$. Since this intersection contains a nonzero homogeneous element, $\sigma\gamma = \sigma\gamma'$, and therefore $\gamma = \gamma'$. $\square$

Example 2.3. Let k be a field, $\Gamma = S_3 = \langle \sigma, \eta \mid \sigma^3 = \eta^2 = e, \eta\sigma\eta = \sigma^{-1} \rangle$, and $R = kS_3$ the group algebra with its natural S_3 -grading $R_\tau = k\tau$. Then $\Gamma_R = S_3$ and $C_{S_3}(\Gamma_R) = \{e\}$. Take $\gamma = \eta \notin C_{S_3}(\Gamma_R)$ and $\delta = 0$. Suppose for a contradiction that $R[t; 0]$ admits a good grading with $\deg(t) = \eta$. Take $x = \sigma \in R_\sigma$. By condition (i), $x \in R[t; 0]_\sigma$, and since $t \in R[t; 0]_\eta$, the grading axiom gives

$$tx \in R[t; 0]_{\eta\sigma}, \quad xt \in R[t; 0]_{\sigma\eta}.$$

Condition (ii) requires the relation $tx = xt$ to be homogeneous, that is, both sides must have the same degree. But $\eta\sigma \neq \sigma\eta$ in S_3 , so tx and xt have distinct degrees, contradicting condition (ii). Hence no good grading exists.

The preceding example uses the zero derivation. The next shows that the centralizer condition cannot be dropped even for a nonzero homogeneous derivation.

Example 2.4 (A nonzero derivation and the centralizer condition). Let $\mathbb{F}$ be a field and let $\Gamma = S_3 = \langle \sigma, \eta \mid \sigma^3 = \eta^2 = e, \eta\sigma\eta = \sigma^{-1} \rangle$. Let $V = \mathbb{F}v \oplus \mathbb{F}w$ be a two-dimensional graded $\mathbb{F}$ -vector space with $\deg(v) = \sigma$, $\deg(w) = \sigma\eta$, and

let $R := \mathbb{F} \oplus V$ be the square-zero extension, so $V^2 = 0$, with

$$R_e = \mathbb{F}, \quad R_\sigma = \mathbb{F}v, \quad R_{\sigma\eta} = \mathbb{F}w,$$

and all other homogeneous components zero. Define the $\mathbb{F}$ -linear map $\delta : R \rightarrow R$ by

$$\delta(1) = 0, \quad \delta(v) = w, \quad \delta(w) = 0.$$

Then δ is a nonzero η -derivation, since $\delta(R_\tau) \subseteq R_{\tau\eta}$ for every $\tau \in \Gamma_R$. However, $\eta \notin C_\Gamma(\Gamma_R)$ because $\eta\sigma \neq \sigma\eta$.

Suppose that $R[t; \delta]$ admitted a good grading with $\deg(t) = \gamma'$. Since v and w are homogeneous, the relation $tv - vt = \delta(v) = w$ forces $\gamma'\sigma = \sigma\gamma'$ and $\gamma'\sigma = \sigma\eta$. Hence $\gamma' = \eta$, and therefore $\eta\sigma = \sigma\eta$, a contradiction. Thus $R[t; \delta]$ admits no good grading.

The following remark gives another interpretation of the centralizer condition in Proposition 2.1.

Remark 2.5. For $\gamma \in \Gamma$, let $R(\gamma)$ denote the additive group R with shifted grading

$$R(\gamma)_\sigma := R_{\sigma\gamma}.$$

We equip $R(\gamma)$ with the left and right R -actions induced by multiplication in R . The left action is always graded, since $R_\sigma R(\gamma)_\tau \subseteq R(\gamma)_{\sigma\tau}$. The right action is graded if and only if $\gamma \in C_\Gamma(\Gamma_R)$.

Indeed, if γ centralizes Γ_R , then $R(\gamma)_\sigma R_\tau \subseteq R(\gamma)_{\sigma\tau}$. Conversely, suppose that the right action is graded and let $\tau \in \Gamma_R$. Since $1_R \in R(\gamma)_{\gamma^{-1}} = R_e$, we have

$$R_\tau = 1_R R_\tau \subseteq R(\gamma)_{\gamma^{-1}\tau} = R_{\gamma^{-1}\tau\gamma}.$$

As $R_\tau \neq 0$, it follows that $\gamma^{-1}\tau\gamma = \tau$. Thus $\gamma \in C_\Gamma(\Gamma_R)$.

Suppose now that this condition holds, and form the square-zero extension

$$T := R \ltimes R(\gamma), \quad (r, m)(r', m') = (rr', rm' + mr'),$$

with $T_\sigma := R_\sigma \oplus R(\gamma)_\sigma$. Then T is a Γ -graded ring.

If $\delta : R \rightarrow R$ is additive, define $\varphi_\delta(x) := (x, \delta(x))$. A direct computation shows that $\varphi_\delta : R \rightarrow T$ is a ring homomorphism if and only if δ is a derivation. Moreover, φ_δ is graded if and only if $\delta(R_\sigma) \subseteq R_{\sigma\gamma}$ for every $\sigma \in \Gamma_R$. Hence the γ -derivations of R are naturally identified with the graded ring sections of the projection $T \rightarrow R, (r, m) \mapsto r$.

We now establish the universal property of $R[t; \delta]$, which underlies the structural isomorphisms of Corollaries 2.7–2.9.

Theorem 2.6 (Universal property). *Let R be a graded ring and $\delta \in \text{Der}_\gamma(R)$. The pair $(R[t; \delta], t)$ is the initial object of C_δ : For every graded ring S , graded ring homomorphism $\phi : R \rightarrow S$, and homogeneous element $u \in S_\gamma$ satisfying (1.8), there exists a unique graded ring homomorphism $\hat{\phi} : R[t; \delta] \rightarrow S$ making the diagram*

$$\begin{array}{ccc} R & \xhookrightarrow{\iota} & R[t; \delta] \\ & \searrow \phi & \downarrow \exists! \hat{\phi} \\ & & S \end{array}$$

commute and satisfying $\hat{\phi}(t) = u$.

Proof. Any ring homomorphism $R[t; \delta] \rightarrow S$ extending ϕ with $t \mapsto u$ must satisfy $\psi(x_i t^i) = \phi(x_i) u^i$ for all $x_i \in R$ and $i \geq 0$, so there is at most one such map, namely

$$\hat{\phi}\left(\sum_{i \geq 0} x_i t^i\right) := \sum_{i \geq 0} \phi(x_i) u^i.$$

This formula defines an additive map. Since $R[t; \delta]$ is generated by R and t subject only to (1.1), multiplicativity reduces to verifying compatibility with that relation. For any $x \in R$,

$$\hat{\phi}(tx) = u \phi(x) = \phi(x) u + \phi(\delta(x)) = \hat{\phi}(xt + \delta(x)),$$

where the middle equality is (1.8), so $\hat{\phi}$ is a ring homomorphism. It is graded: For $x_i \in R_{\tau\gamma-i}$, one has $\phi(x_i) \in S_{\tau\gamma-i}$ and $u^i \in S_{\gamma i}$, hence $\phi(x_i) u^i \in S_{\tau}$. $\square$

The universal property yields the following structural isomorphisms, which together prove Theorem 1.9.

Corollary 2.7. *Let $\delta_1, \delta_2 \in \text{Der}_{\gamma}(R)$ and suppose $\delta_2 = \alpha \circ \delta_1 \circ \alpha^{-1}$ for some $\alpha \in \text{Aut}_{\text{gr}}(R)$. There is a graded isomorphism $\tilde{\alpha} : R[t; \delta_1] \xrightarrow{\sim} R[t; \delta_2]$ extending α .*

Proof. For any $x \in R$, the commutation rule in $R[t; \delta_2]$ gives $t \alpha(x) - \alpha(x) t = \alpha(\delta_1(x))$, so (1.8) holds in $R[t; \delta_2]$ with $\phi = \alpha$ and $u = t$. Theorem 2.6 produces a unique graded ring homomorphism $\tilde{\alpha} : R[t; \delta_1] \rightarrow R[t; \delta_2]$ extending α with $\tilde{\alpha}(t) = t$. Applying the same argument to α^{-1} yields an inverse, so $\tilde{\alpha}$ is an isomorphism. $\square$

Corollary 2.8. *Let $\delta = \delta_1 + \delta_2$ where $\delta_1, \delta_2 \in \text{Der}_{\gamma}(R)$ and $\delta_1 = \delta_r$ is the inner derivation $\delta_r(x) := rx - xr$ for some $r \in R_{\gamma}$. Then $R[t; \delta] \cong_{\text{gr}} R[t; \delta_2]$.*

Proof. Set $u := t - r \in R[t; \delta]$. Since $r \in R_{\gamma}$ and $t \in R[t; \delta]_{\gamma}$, the element u is homogeneous of degree γ . For any $x \in R$,

$$ux - xu = (t - r)x - x(t - r) = tx - xt - (rx - xr) = \delta(x) - \delta_r(x) = \delta_2(x),$$

so $(R[t; \delta], u)$ is an object of C_{δ_2} . By Theorem 2.6, there is a unique graded ring homomorphism $\varphi : R[t; \delta_2] \rightarrow R[t; \delta]$ with $\varphi|_R = \text{id}$ and $\varphi(t) = t - r$.

For the inverse, set $v := t + r \in R[t; \delta_2]$. Since $\delta_r(x) = rx - xr$, a direct computation gives $vx - xv = \delta(x)$, so $(R[t; \delta_2], v)$ is an object of C_{δ} . Theorem 2.6 gives a graded ring homomorphism $\psi : R[t; \delta] \rightarrow R[t; \delta_2]$ with $\psi|_R = \text{id}$ and $\psi(t) = t + r$. Both $\psi \circ \varphi$ and $\varphi \circ \psi$ fix R pointwise and fix t , so uniqueness in Theorem 2.6 gives $\psi \circ \varphi = \text{id}$ and $\varphi \circ \psi = \text{id}$. $\square$

Corollary 2.9. *Let T be a graded ring, $\delta \in \text{Der}_{\gamma}(T)$, and $R := M_n(T)$ with the grading $M_n(T)_{\tau} := M_n(T_{\tau})$. Let $\hat{\delta}$ be the entrywise γ -derivation $\hat{\delta}((x_{ij})) := (\delta(x_{ij}))$. Then $M_n(T)[t; \hat{\delta}] \cong_{\text{gr}} M_n(T[t; \delta])$.*

Proof. Let $\phi : M_n(T) \rightarrow M_n(T[t; \delta])$ be the map sending (x_{ij}) to the same matrix with entries in $T[t; \delta]$. This is a graded ring homomorphism since $T \hookrightarrow T[t; \delta]$ is graded. Set $u := tI_n \in M_n(T[t; \delta])_{\gamma}$. The relation $tx_{ij} - x_{ij}t = \delta(x_{ij})$ in $T[t; \delta]$ gives, entrywise,

$$u \phi(A) - \phi(A) u = (\delta(x_{ij}))_{ij} = \phi(\hat{\delta}(A))$$

for any $A = (x_{ij}) \in M_n(T)$, so (1.8) holds and Theorem 2.6 yields a unique graded ring homomorphism $\varphi : M_n(T)[t; \hat{\delta}] \rightarrow M_n(T[t; \delta])$ with $\varphi|_{M_n(T)} = \phi$ and $\varphi(t) = tI_n$.

Both rings are free left $M_n(T)$ -modules on the basis $\{I_n, t, t^2, \dots\}$, and φ acts by $\varphi(\sum_k A_k t^k) = \sum_k A_k t^k$. Surjectivity is immediate. If $\sum_k A_k t^k \in \ker \varphi$, then each $A_k = 0$ by uniqueness of coefficients, so φ is injective. $\square$

For a nonzero element

$$f = \sum_{i=0}^n x_i t^i \in R[t; \delta], \quad x_n \neq 0,$$

we write $\deg_t(f) := n$ for its t -degree and $\text{lc}(f) := x_n$ for its leading coefficient. For a graded ideal $\mathfrak{a} \subseteq R[t; \delta]$ and $n \geq 0$, the n -th leading-coefficient ideal is

$$\mathcal{L}_n(\mathfrak{a}) := \{0\} \cup \{x \in R : x \text{ is the leading coefficient of some element of } \mathfrak{a} \text{ of } t\text{-degree } n\}.$$

Lemma 2.10. *Let $\mathfrak{a} \subseteq R[t; \delta]$ be a graded ideal and $n \geq 0$. Then $\mathcal{L}_n(\mathfrak{a})$ is a δ -invariant graded ideal of R .*

Proof. We first show that $\mathcal{L}_n(\mathfrak{a})$ is graded. Let $x \in \mathcal{L}_n(\mathfrak{a})$, witnessed by $f = \sum_{i=0}^n a_i t^i \in \mathfrak{a}$ with $a_n = x$. Write $a_i = \sum_{\sigma \in \Gamma_R} (a_i)_\sigma$ for the homogeneous decomposition of each coefficient in R . By (1.4), the τ -homogeneous component of f in $R[t; \delta]$ is

$$f_\tau = \sum_{i=0}^n (a_i)_{\tau\gamma^{-i}} t^i. \quad (2.11)$$

Since $\mathfrak{a}$ is a graded ideal, $f_\tau \in \mathfrak{a}$ for every $\tau \in \Gamma$. Let $\sigma \in \Gamma_R$ with $x_\sigma \neq 0$; setting $\tau = \sigma\gamma^n$ in (2.11) shows that $f_{\sigma\gamma^n} \in \mathfrak{a}$ has t -degree n with leading coefficient $(a_n)_\sigma = x_\sigma$. Hence $x_\sigma \in \mathcal{L}_n(\mathfrak{a})$. Every homogeneous component of x thus lies in $\mathcal{L}_n(\mathfrak{a})$, giving

$$\mathcal{L}_n(\mathfrak{a}) = \bigoplus_{\sigma \in \Gamma_R} (\mathcal{L}_n(\mathfrak{a}) \cap R_\sigma),$$

so $\mathcal{L}_n(\mathfrak{a})$ is graded.

We now verify the ideal property on homogeneous elements, which suffices since both R and $\mathcal{L}_n(\mathfrak{a})$ are graded. Let $x \in \mathcal{L}_n(\mathfrak{a}) \cap R_\sigma$, witnessed by $f = xt^n + \dots \in \mathfrak{a}_{\sigma\gamma^n}$, and let $r \in R_\tau$. For left multiplication, $rf \in \mathfrak{a}_{\tau\sigma\gamma^n}$ has t -degree n with leading coefficient $rx \in R_{\tau\sigma}$, so $rx \in \mathcal{L}_n(\mathfrak{a})$. For right multiplication, (1.2) gives

$$fr = xr t^n + (\text{terms of } t\text{-degree} < n);$$

since $\gamma \in C_\Gamma(\Gamma_R)$, the product fr is homogeneous of degree $\sigma\tau\gamma^n$, so $fr \in \mathfrak{a}_{\sigma\tau\gamma^n}$ and $xr \in \mathcal{L}_n(\mathfrak{a})$.

For δ -invariance, since $\mathfrak{a}$ is a two-sided ideal, both tf and ft lie in $\mathfrak{a}$, and a direct computation gives

$$tf - ft = [xt^{n+1} + (\delta(x) + a_{n-1})t^n + \dots] - [xt^{n+1} + x_{n-1}t^n + \dots] = \delta(x)t^n + \dots,$$

so $tf - ft \in \mathfrak{a}$ has t^n -coefficient $\delta(x)$, whence $\delta(x) \in \mathcal{L}_n(\mathfrak{a})$. □

Lemma 2.12. *Let R be a δ -gr-simple ring of characteristic zero. Then R is torsion-free as a $\mathbb{Z}$ -module and every integer $n \geq 1$ is invertible in R . In particular, the canonical map $\mathbb{Z} \rightarrow R$ extends uniquely to an embedding $\mathbb{Q} \hookrightarrow Z(R)$, so R is a $\mathbb{Q}$ -algebra.*

Proof. Let

$$\mathfrak{t} := \{x \in R : \text{there exists } n \geq 1 \text{ with } nx = 0\}.$$

The set $\mathfrak{t}$ is a two-sided graded ideal, and it is δ -invariant because $n\delta(x) = \delta(nx) = 0$. Since $\text{char } R = 0$, one has $1_R \notin \mathfrak{t}$, so δ -gr-simplicity gives $\mathfrak{t} = 0$. Thus R is torsion-free as a $\mathbb{Z}$ -module.

Now fix $n \geq 1$. The ideal

$$nR = (n \cdot 1_R)R$$

is nonzero, graded, two-sided, and δ -invariant, since $n \cdot 1_R \in R_e \cap Z(R)$ and $\delta(nr) = n\delta(r)$. Hence $nR = R$. Thus $n \cdot 1_R$ is invertible in R . Since it is central, its inverse is also central, and the stated embedding of $\mathbb{Q}$ follows. $\square$

Lemma 2.13. *Let R be a gr-simple ring of characteristic zero and $\delta \in \text{Der}_\gamma(R)$ a γ -outer derivation. If $s = \sum_{i=0}^n x_i t^i \in C_{R[t;\delta]}(R)$ and there exists $\sigma \in Z(\Gamma)$ with $(x_n)_\sigma \neq 0$, then $n = 0$ and $s \in Z(R)$.*

Proof. Suppose $n > 0$ and let $c \in \mathcal{H}(R)$. Using (1.2),

$$cs = \sum_{i=0}^n cx_i t^i, \quad sc = \sum_{i=0}^n x_i t^i c = \sum_{i=0}^n \sum_{k=0}^i \binom{i}{k} x_i \delta^k(c) t^{i-k}.$$

Comparing the t^n -coefficients of $cs = sc$ gives $cx_n = x_n c$ for all homogeneous $c \in R$, so $x_n \in Z(R)$. Since $\sigma \in Z(\Gamma)$, it follows that $(x_n)_\sigma R$ is a graded ideal of R ; gr-simplicity then forces $(x_n)_\sigma \in R^\times$. Comparing the t^{n-1} -coefficients of $cs = sc$ gives

$$cx_{n-1} - x_{n-1}c = n x_n \delta(c) \quad \text{for all } c \in \mathcal{H}(R). \quad (2.14)$$

For $c \in R_\tau$, we have $\delta(c) \in R_{\tau\gamma}$. Taking the $R_{\sigma\tau\gamma}$ -component of (2.14) and using $\sigma, \gamma \in C_\Gamma(\Gamma_R)$, we obtain

$$c(x_{n-1})_{\sigma\gamma} - (x_{n-1})_{\sigma\gamma}c = n(x_n)_\sigma \delta(c).$$

Since R is gr-simple, it is in particular δ -gr-simple. Lemma 2.12 therefore shows that $n \cdot 1_R$ is invertible. Since $(x_n)_\sigma^{-1} \in Z(R)$, left-multiplying by $(n \cdot 1_R)^{-1}(x_n)_\sigma^{-1}$ yields

$$\delta(c) = r c - c r = \delta_r(c),$$

where $r := -(n \cdot 1_R)^{-1}(x_n)_\sigma^{-1}(x_{n-1})_{\sigma\gamma}$. Moreover, $r \in R_\gamma$ since $(x_n)_\sigma^{-1} \in R_{\sigma^{-1}}$ and $(x_{n-1})_{\sigma\gamma} \in R_{\sigma\gamma}$, hence $r \in R_{\sigma^{-1}\sigma\gamma} = R_\gamma$. Thus $\delta = \delta_r$ with $r \in R_\gamma$, contradicting the assumption that δ is γ -outer. Therefore $n = 0$, and $s = x_0 \in Z(R)$. $\square$

3 | Graded Ring-Theoretic Properties of $R[t; \delta]$

We begin the analysis of ideal-theoretic properties of $R[t; \delta]$ by reducing graded-ideal questions to suitable ideals of the coefficient ring via leading-coefficient ideals. We then treat gr-simplicity, gr-primality, and gr-Noetherianity in separate subsections.

3.1 | Gr-Simplicity

Proposition 3.1. *Let R be a graded ring of characteristic zero and $\delta \in \text{Der}_\gamma(R)$. If R is gr-simple and δ is γ -outer, then $R[t; \delta]$ is gr-simple.*

Proof. Let $\mathfrak{a}$ be a nonzero graded ideal of $R[t; \delta]$, and set $n := \min\{\deg_t(f) : f \in \mathfrak{a}, f \neq 0\}$. By Lemma 2.10, $\mathcal{L}_n(\mathfrak{a})$ is a nonzero graded ideal of R , so gr-simplicity gives $\mathcal{L}_n(\mathfrak{a}) = R$. In particular $1 \in R_e \subseteq \mathcal{L}_n(\mathfrak{a})$, so there exists $g \in \mathfrak{a}$ of t -degree n with leading coefficient 1. Write $g = \sum_{\tau \in \Gamma} g_\tau$ with $g_\tau \in R[t; \delta]_\tau$. Since $\mathfrak{a}$ is a graded ideal, $g_\tau \in \mathfrak{a}_\tau$ for every τ . By (1.4), the

t^n -coefficient of g_τ lies in $R_{\tau\gamma-n}$; since the overall leading coefficient of g is $1 \in R_e$, the only component g_τ with nonzero t^n -coefficient is g_{γ^n} , whose t^n -coefficient is $1 \in R_e$. Set $f := g_{\gamma^n} \in \mathfrak{a}_{\gamma^n}$; then f is monic of t -degree n with leading coefficient $1 \in R_e$.

For any $c \in \mathcal{H}(R)$, (1.2) gives

$$t^n c = c t^n + n \delta(c) t^{n-1} + \sum_{k=2}^n \binom{n}{k} \delta^k(c) t^{n-k},$$

so

$$\begin{aligned} f c - c f &= (t^n + x_{n-1} t^{n-1} + \cdots + x_0) c - c (t^n + x_{n-1} t^{n-1} + \cdots + x_0) \\ &= (n \delta(c) + x_{n-1} c - c x_{n-1}) t^{n-1} + (\text{terms of } t\text{-degree} < n-1). \end{aligned}$$

Since $\mathfrak{a}$ is two-sided, $f c - c f \in \mathfrak{a}$, and $\deg_t(f c - c f) \leq n-1 < n$, so minimality of n forces $f c - c f = 0$. Hence $f \in C_{R[t;\delta]}(R)$. The leading coefficient of f is $1 \in R_e$. Since $e \in Z(\Gamma)$ and $(1)_e = 1 \neq 0$, Lemma 2.13 applies and gives $n = 0$, hence $f = 1 \in \mathfrak{a}$ and $\mathfrak{a} = R[t;\delta]$. $\square$

Proposition 3.2. *Let R be a graded ring of characteristic zero and $\delta \in \text{Der}_\gamma(R)$. If R is δ -gr-simple and δ is γ -outer, then $R[t;\delta]$ is gr-simple.*

Proof. Let $\mathfrak{a}$ be a nonzero graded ideal of $R[t;\delta]$, and let n be the minimal t -degree of a nonzero element of $\mathfrak{a}$. By Lemma 2.10, $\mathcal{L}_n(\mathfrak{a})$ is a nonzero δ -invariant graded ideal of R . Since R is δ -gr-simple, we have $\mathcal{L}_n(\mathfrak{a}) = R$, and hence $1 \in R_e \subseteq \mathcal{L}_n(\mathfrak{a})$. By the same argument as in the proof of Proposition 3.1, there exists a monic element $f = t^n + x_{n-1} t^{n-1} + \cdots + x_0 \in \mathfrak{a}_{\gamma^n}$ of t -degree n with leading coefficient $1 \in R_e$. As in that proof, we obtain

$$f c - c f = 0 \quad \text{for all } c \in R,$$

and hence

$$n \delta(c) + x_{n-1} c - c x_{n-1} = 0 \quad \text{for all } c \in R.$$

If $n > 0$, then Lemma 2.12 shows that $n \cdot 1_R$ is invertible. Since $x_{n-1} \in R_{\gamma^n \gamma^{-(n-1)}} = R_\gamma$, we obtain

$$\delta(c) = c r - r c,$$

with $r = -(n \cdot 1_R)^{-1} x_{n-1} \in R_\gamma$, contradicting the assumption that δ is γ -outer. Therefore $n = 0$, hence $f = 1 \in \mathfrak{a}$, and $\mathfrak{a} = R[t;\delta]$. $\square$

Proposition 3.3. *Let R be a graded ring and $\delta \in \text{Der}_\gamma(R)$. If $R[t;\delta]$ is gr-simple, then R is δ -gr-simple and δ is γ -outer.*

Proof. Let $\mathfrak{a} \subsetneq R$ be a nonzero δ -invariant graded ideal. The set

$$\mathfrak{a}[t;\delta] := \left\{ \sum_{i \geq 0} x_i t^i : x_i \in \mathfrak{a} \right\} \subseteq R[t;\delta]$$

is closed under addition, under left and right multiplication by elements of R (since $\mathfrak{a}$ is a two-sided ideal of R), and under left multiplication by t since $t x = x t + \delta(x)$ with $x, \delta(x) \in \mathfrak{a}$ by δ -invariance. It is graded: For any $f = \sum_{i \geq 0} x_i t^i \in \mathfrak{a}[t;\delta]$, the τ -homogeneous component of f in $R[t;\delta]$ is $f_\tau = \sum_{i \geq 0} (x_i)_{\tau\gamma^{-i}} t^i$ by (1.4); since $\mathfrak{a}$ is a graded ideal of R , each $(x_i)_{\tau\gamma^{-i}} \in \mathfrak{a}$, so $f_\tau \in \mathfrak{a}[t;\delta]$. Hence $\mathfrak{a}[t;\delta]$ is a nonzero graded ideal of $R[t;\delta]$. Since $1 \notin \mathfrak{a}$, it is proper, contradicting gr-simplicity of $R[t;\delta]$.

Suppose now that $\delta = \delta_r$ for some $r \in R_\gamma$. Since $rx - xr = \delta(x)$ for all $x \in R$, Theorem 2.6 yields a graded ring homomorphism $\text{ev}_r : R[t; \delta] \rightarrow R$ with $\text{ev}_r|_R = \text{id}$ and $\text{ev}_r(t) = r$. Then $t - r \in \ker(\text{ev}_r)$ and $t - r \neq 0$, so $\ker(\text{ev}_r)$ is a proper nonzero graded ideal of $R[t; \delta]$, contradicting gr-simplicity. $\square$

Theorem 1.10(2) follows from Propositions 3.2 and 3.3.

Corollary 3.4. *If $\text{char } R = 0$, then $R[t; \delta]$ is gr-simple if and only if R is δ -gr-simple and δ is γ -outer.*

The following example shows that the condition $\gamma \in C_\Gamma(\Gamma_R)$ is genuinely weaker than requiring $\gamma \in Z(\Gamma)$, and that the gr-simplicity criterion applies in this situation.

Example 3.5. Let $\mathbb{F}$ be a field of characteristic zero and let $\Gamma = D_\infty = \langle a, s \mid s^2 = e, sas^{-1} = a^{-1} \rangle$. Set $R = M_2(\mathbb{F})[x, x^{-1}]$, where x is central, and grade R by $R_{ak} = M_2(\mathbb{F})x^k$ for $k \in \mathbb{Z}$, with $R_\tau = 0$ for $\tau \notin \langle a \rangle$. Then $\Gamma_R = \langle a \rangle$ and

$$C_\Gamma(\Gamma_R) = \langle a \rangle, \quad Z(\Gamma) = \{e\}.$$

In particular, $\gamma := a^{-1}$ belongs to $C_\Gamma(\Gamma_R) \setminus Z(\Gamma)$.

Define $\delta : R \rightarrow R$ by $\delta(Mx^k) = kMx^{k-1}$ for $M \in M_2(\mathbb{F})$ and $k \in \mathbb{Z}$. This is a derivation, and $\delta(R_{ak}) \subseteq R_{ak-1} = R_{ak\gamma}$, so $\delta \in \text{Der}_\gamma(R)$.

We claim that R is gr-simple. Indeed, let $\mathfrak{a}$ be a nonzero graded ideal and choose a nonzero homogeneous element $Mx^k \in \mathfrak{a}$. Since $M_2(\mathbb{F})$ is simple, there exist $A_i, B_i \in M_2(\mathbb{F})$ such that $\sum_i A_i M B_i = I_2$. Hence

$$I_2 = \sum_i A_i (Mx^k) (B_i x^{-k}) \in \mathfrak{a},$$

and therefore $\mathfrak{a} = R$. In particular, R is δ -gr-simple.

Finally, δ is γ -outer. Indeed, $xI_2 \in Z(R)$, whereas $\delta(xI_2) = I_2 \neq 0$. Since every inner derivation vanishes on $Z(R)$, the derivation δ cannot be inner. Thus, by Theorem 1.10(2), $R[t; \delta]$ is gr-simple.

The following two propositions give the center criterion appearing in Theorem 1.10(3).

Proposition 3.6. *Let R be a δ -gr-simple ring and $\delta \in \text{Der}_\gamma(R)$. Every nonzero graded ideal $\mathfrak{a}$ of $R[t; \delta]$ satisfies $\mathfrak{a} \cap Z(R[t; \delta]) \neq 0$.*

Proof. Set $n := \min\{\deg_t(f) : f \in \mathfrak{a}, f \neq 0\}$. By Lemma 2.10, $\mathcal{L}_n(\mathfrak{a})$ is a nonzero δ -invariant graded ideal of R , so δ -gr-simplicity gives $\mathcal{L}_n(\mathfrak{a}) = R$. By the argument of Proposition 3.2, there exists a monic element

$$f = t^n + x_{n-1}t^{n-1} + \cdots + x_0 \in \mathfrak{a}_{\gamma^n}$$

with $f \in C_{R[t; \delta]}(R)$. It remains to show $ft = tf$. We have

$$ft = t^{n+1} + \sum_{i=0}^{n-1} x_i t^{i+1}, \quad tf = t^{n+1} + \sum_{i=0}^{n-1} (x_i t^{i+1} + \delta(x_i) t^i).$$

So $ft - tf = -\sum_{i=0}^{n-1} \delta(x_i) t^i \in \mathfrak{a}$. Since $\deg_t(ft - tf) \leq n-1 < n$, minimality gives $ft = tf$. Since R and t generate $R[t; \delta]$, it follows that $f \in Z(R[t; \delta])$. $\square$

Proposition 3.7. *Let R be a graded ring and $\delta \in \text{Der}_\gamma(R)$. Assume that $Z(R[t; \delta])$ is a graded subring of $R[t; \delta]$. Then the following are equivalent.*

- (1) $R[t; \delta]$ is gr-simple.
- (2) R is δ -gr-simple and $Z(R[t; \delta])$ is a graded field.

Proof. (1) $\Rightarrow$ (2). Proposition 3.3 gives δ -gr-simplicity of R . For any nonzero homogeneous $f \in Z(R[t; \delta])$, the ideal $fR[t; \delta]$ is a nonzero graded ideal of $R[t; \delta]$, so $fR[t; \delta] = R[t; \delta]$ and $f \in R[t; \delta]^\times$. Hence $Z(R[t; \delta])$ is a graded field.

(2) $\Rightarrow$ (1). Let $\mathfrak{a}$ be a nonzero graded ideal of $R[t; \delta]$. By Proposition 3.6, $\mathfrak{a} \cap Z(R[t; \delta]) \neq 0$; pick a nonzero homogeneous $f \in \mathfrak{a} \cap Z(R[t; \delta])$. Since $Z(R[t; \delta])$ is a graded field, f is invertible, so $1 \in \mathfrak{a}$ and $\mathfrak{a} = R[t; \delta]$. $\square$

Remark 3.8. The hypothesis in Proposition 3.7 concerns the grading on the center, not the ordinary field property of the center. For a general group grading, $Z(R[t; \delta])$ need not be a graded subring, so the phrase “ $Z(R[t; \delta])$ is a graded field” requires this point to be addressed. The hypothesis is automatic when Γ is abelian [18, Lemma 2.2]. More generally, [19, Proposition 2.1] applies when Γ admits a total order invariant under left and right multiplication and every nonzero homogeneous element of $R[t; \delta]$ is regular. Under either of these conditions, Proposition 3.7 gives the characteristic-free graded analogue of the Oinert–Silvestrov criterion.

The following corollary describes how gr-simplicity behaves under the isomorphisms of Corollaries 2.7–2.9, proving Theorem 1.10(4).

Corollary 3.9.

- (1) If δ_1 and δ_2 are conjugate γ -derivations, then $R[t; \delta_1]$ is gr-simple if and only if $R[t; \delta_2]$ is.
- (2) If $\delta = \delta_r + \delta_2$ with $r \in R_\gamma$ and $\delta_2 \in \text{Der}_\gamma(R)$, then $R[t; \delta]$ is gr-simple if and only if $R[t; \delta_2]$ is.
- (3) Let T be a graded ring, $\delta \in \text{Der}_\gamma(T)$, and $S := M_n(T)$ with the induced grading and entrywise γ -derivation $\hat{\delta}$. Then $S[t; \hat{\delta}]$ is gr-simple if and only if $T[t; \delta]$ is.

Proof. Parts (1) and (2) follow from Corollaries 2.7 and 2.8, since gr-simplicity is preserved under graded isomorphisms. For (3), Corollary 2.9 gives $M_n(T)[t; \hat{\delta}] \cong_{\text{gr}} M_n(T[t; \delta])$. For any graded ring B , the map $\mathfrak{b} \mapsto M_n(\mathfrak{b})$ is a lattice isomorphism between the graded ideals of B and those of $M_n(B)$. Hence $M_n(B)$ is gr-simple if and only if B is. Applying this with $B = T[t; \delta]$ gives the result. $\square$

One may ask whether the construction $R \mapsto R[t; \delta]$ preserves gr-simplicity under iteration. The first step already fails: $R[t]$ is never gr-simple, since $tR[t]$ is a proper graded ideal. The derivation $\delta := \frac{d}{dt}$ repairs this at the second step. This is Theorem 1.10(5).

Proposition 3.10. Let R be a gr-simple graded ring of characteristic zero, and let $R' := R[t; 0]$ with $R'_t := R_t[t]$ and $\deg(t) = e$. Then $R'[s; \frac{d}{dt}]$ is gr-simple.

Proof. Let $\delta := \frac{d}{dt} : R' \rightarrow R'$ denote the standard derivation. The defining relation in $R'[s; \delta]$ is $sx = xs + \delta(x)$ for all $x \in R'$. In particular, $st = ts + 1$ and $sx = xs$ for all $x \in R$.

Let $\mathfrak{a}$ be a nonzero graded ideal of $R'[s; \delta]$ and let $f \in \mathfrak{a}$ be a nonzero element. Write

$$f = \sum_{i=0}^n t^i p_i(s),$$

where $p_i(s) = \sum_j x_{ij} s^j \in R[s]$ and $p_n(s) \neq 0$. For any $x \in R$ and $k \geq 0$, the defining relation gives $[s, t^k x] = \delta(t^k x) = k t^{k-1} x$, and $[s, p_i(s)] = 0$ since δ vanishes on R and s commutes with itself. Therefore, setting $\text{ad}_s(f) := [s, f]$,

$$\text{ad}_s^n(t^i p_i(s)) = \begin{cases} n! p_n(s) & \text{if } i = n, \\ 0 & \text{if } i < n. \end{cases}$$

Since $\mathfrak{a}$ is a two-sided ideal and $s \in R'[s; \delta]$, we have $\text{ad}_s^n(f) = n! p_n(s) \in \mathfrak{a}$. As $\text{char}(R) = 0$, the element $n! \cdot 1_R$ is invertible in R , so $p_n(s) \in \mathfrak{a}$.

Write $p_n(s) = \sum_{j=0}^m x_j s^j$ with $x_j \in R$ and $x_m \neq 0$. From $st = ts + 1$ we get $[t, s] = -1$, and since $[t, x] = 0$ for all $x \in R$, the Leibniz rule gives $[t, s^j] = -j s^{j-1}$, hence $\text{ad}_t(x_j s^j) = -j x_j s^{j-1}$. Iterating m times,

$$\text{ad}_t^m(p_n(s)) = (-1)^m m! x_m.$$

Since $p_n(s) \in \mathfrak{a}$ and $t \in R'[s; \delta]$, we have $\text{ad}_t^m(p_n(s)) \in \mathfrak{a}$, and since $m!$ is invertible in R , it follows that $x_m \in \mathfrak{a}$. The element x_m is nonzero and belongs to R_σ for some $\sigma \in \Gamma_R$, so $Rx_m R$ is a nonzero graded ideal of R . Since R is gr-simple, $Rx_m R = R$, hence $1_R \in \mathfrak{a}$, and therefore $\mathfrak{a} = R'[s; \frac{d}{dt}]$. $\square$

Remark 3.11. Alternatively, one can verify directly that $\delta = \frac{d}{dt}$ is e -outer on R' and that R' is δ -gr-simple, and then conclude by Corollary 3.4.

3.2 | Gr-Primality

Proposition 3.12. *Let R be a δ -gr-prime graded ring and $\delta \in \text{Der}_\gamma(R)$. Then $R[t; \delta]$ is gr-prime.*

Proof. Suppose for a contradiction that $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero graded ideals of $R[t; \delta]$ with $\mathfrak{a}\mathfrak{b} = 0$. Let $n := \min\{\deg_t(f) : f \in \mathfrak{a}, f \neq 0\}$ and $m := \min\{\deg_t(g) : g \in \mathfrak{b}, g \neq 0\}$. By Lemma 2.10, $\mathcal{L}_n(\mathfrak{a})$ and $\mathcal{L}_m(\mathfrak{b})$ are nonzero δ -invariant graded ideals of R . Let $x \in \mathcal{L}_n(\mathfrak{a})_\sigma$ and $x' \in \mathcal{L}_m(\mathfrak{b})_\tau$ be homogeneous, and pick

$$f = xt^n + x_{n-1}t^{n-1} + \cdots + x_0 \in \mathfrak{a}_{\sigma\gamma^n}, \quad g = x't^m + x'_{m-1}t^{m-1} + \cdots + x'_0 \in \mathfrak{b}_{\tau\gamma^m}.$$

Since $\mathfrak{a}\mathfrak{b} = 0$, we have $fg = 0$. By (1.2),

$$(xt^n)(x't^m) = xx't^{n+m} + \sum_{k=1}^n \binom{n}{k} x \delta^k(x') t^{n+m-k}.$$

Every other summand $(x_i t^i)(x'_j t^j)$ with $i < n$ or $j < m$ contributes only to t -degrees strictly less than $n + m$, so the t^{n+m} -coefficient of fg is xx' . Since $fg = 0$, we get $xx' = 0$. As $x \in \mathcal{L}_n(\mathfrak{a})$ and $x' \in \mathcal{L}_m(\mathfrak{b})$ were arbitrary homogeneous elements, $\mathcal{L}_n(\mathfrak{a})\mathcal{L}_m(\mathfrak{b}) = 0$, contradicting δ -gr-primality of R . $\square$

The above proposition proves Theorem 1.11(1) in the stronger form that δ -gr-primality of R suffices; the original statement of Theorem 1.11(1) with gr-prime follows as a special case.

Proposition 3.13. *If $R[t; \delta]$ is gr-prime, then R is δ -gr-prime.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero δ -invariant graded ideals of R with $\mathfrak{a}\mathfrak{b} = 0$. As in Proposition 3.3, the extensions $\mathfrak{a}[t; \delta]$ and $\mathfrak{b}[t; \delta]$ are nonzero graded ideals of $R[t; \delta]$. For any $f = \sum_i x_i t^i \in \mathfrak{a}[t; \delta]$ and $g = \sum_j y_j t^j \in \mathfrak{b}[t; \delta]$, (1.2) gives

$$(x_i t^i)(y_j t^j) = \sum_{k=0}^i \binom{i}{k} x_i \delta^k(y_j) t^{i+j-k}.$$

Since $\mathfrak{b}$ is δ -invariant, $\delta^k(y_j) \in \mathfrak{b}$ for every $k \geq 0$, so $x_i \delta^k(y_j) \in \mathfrak{a}\mathfrak{b} = 0$. Hence $fg = 0$ and $\mathfrak{a}[t; \delta]\mathfrak{b}[t; \delta] = 0$. Since $R[t; \delta]$ is gr-prime, $\mathfrak{a}[t; \delta] = 0$ or $\mathfrak{b}[t; \delta] = 0$, giving $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$. $\square$

Combining Propositions 3.12 and 3.13 gives Theorem 1.11(2).

Corollary 3.14. $R[t; \delta]$ is *gr-prime* if and only if R is δ -*gr-prime*.

3.3 | Gr-Noetherianity

We prove that gr-Noetherianity of R passes to $R[t; \delta]$ (Theorem 1.12). The argument is a graded version of the standard Hilbert basis theorem proof: We introduce right leading-coefficient ideals $\mathcal{L}_n^r(\mathfrak{a})$ and show that any ascending chain of graded right ideals of $R[t; \delta]$ stabilizes by comparing the stabilization indices of the finitely many chains $(\mathcal{L}_k^r(\mathfrak{a}_j))_j$ for $k < s_0$ together with the eventual constancy established in (3.17). Taking $\delta = 0$ recovers the graded Hilbert basis theorem as Corollary 3.18.

For a graded right ideal $\mathfrak{a} \subseteq R[t; \delta]$ and $n \geq 0$, define

$$\mathcal{L}_n^r(\mathfrak{a}) := \{0\} \cup \{x \in R : x \text{ is the leading coefficient of some } f \in \mathfrak{a}\}.$$

Lemma 3.15. Let $\mathfrak{a} \subseteq \mathfrak{b}$ be graded right ideals of $R[t; \delta]$ and $n \geq 0$.

- (1) $\mathcal{L}_n^r(\mathfrak{a})$ is a graded right ideal of R .
- (2) $\mathcal{L}_n^r(\mathfrak{a}) \subseteq \mathcal{L}_n^r(\mathfrak{b})$.
- (3) If $\mathcal{L}_n^r(\mathfrak{a}) = \mathcal{L}_n^r(\mathfrak{b})$ for all $n \geq 0$, then $\mathfrak{a} = \mathfrak{b}$.

Proof. Part (1) follows from Lemma 2.10. For part (2), let $x \in \mathcal{L}_n^r(\mathfrak{a})$. Then x is the leading coefficient of some $f \in \mathfrak{a}$ of t -degree n . Since $\mathfrak{a} \subseteq \mathfrak{b}$, we have $f \in \mathfrak{b}$, whence $x \in \mathcal{L}_n^r(\mathfrak{b})$. For part (3), since $\mathfrak{a} \subseteq \mathfrak{b}$, it suffices to show $\mathfrak{b} \subseteq \mathfrak{a}$. We proceed by induction on $\deg_t(g)$ for $g \in \mathfrak{b}$. If $\deg_t(g) = 0$, then $g \in R$ and g is its own leading coefficient, so $g \in \mathcal{L}_0^r(\mathfrak{b}) = \mathcal{L}_0^r(\mathfrak{a})$. Hence there exists $f \in \mathfrak{a}$ of t -degree 0 with leading coefficient g ; since f has t -degree 0, we have $f = g$, so $g \in \mathfrak{a}$. Suppose the claim holds for all elements of $\mathfrak{b}$ of t -degree less than n , and let $g \in \mathfrak{b}$ with $\deg_t(g) = n$ and leading coefficient x . Then $x \in \mathcal{L}_n^r(\mathfrak{b}) = \mathcal{L}_n^r(\mathfrak{a})$, so there exists $f \in \mathfrak{a}$ of t -degree n with leading coefficient x . The difference $g - f$ lies in $\mathfrak{b}$ and satisfies $\deg_t(g - f) < n$, so the induction hypothesis gives $g - f \in \mathfrak{a}$. Since $f \in \mathfrak{a}$, we conclude $g \in \mathfrak{a}$. $\square$

Proposition 3.16. Let $\delta \in \text{Der}_\gamma(R)$.

- (1) If R is *gr-right-Noetherian*, then so is $R[t; \delta]$.
- (2) If R is *gr-left-Noetherian*, then so is $R[t; \delta]$.

Proof. We prove (1); part (2) follows by replacing graded right ideals with graded left ideals and $\mathcal{L}_n^r$ with its left analogue $\mathcal{L}_n^\ell$ throughout.

Let

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$$

be an ascending chain of graded right ideals of $R[t; \delta]$. We show this chain stabilizes.

We construct an ascending chain of graded right ideals of R by coupling the index of the ideal with the degree of the leading coefficient. For any homogeneous $f = at^s + \cdots \in \mathfrak{a}_s$, the product $ft \in \mathfrak{a}_s$ has t -degree $s + 1$ and leading coefficient a , so $\mathcal{L}_s^r(\mathfrak{a}_s) \subseteq \mathcal{L}_{s+1}^r(\mathfrak{a}_s) \subseteq \mathcal{L}_{s+1}^r(\mathfrak{a}_{s+1})$, the second inclusion by Lemma 3.15(2). Hence

$$\mathcal{L}_1^r(\mathfrak{a}_1) \subseteq \mathcal{L}_2^r(\mathfrak{a}_2) \subseteq \cdots$$

is an ascending chain of graded right ideals of R . Since R is gr-right-Noetherian, there exists $s_0 \geq 1$ such that

$$\mathcal{L}_{s_0}^r(\mathfrak{a}_{s_0}) = \mathcal{L}_{s_0+n}^r(\mathfrak{a}_{s_0+n}) \quad \text{for all } n \geq 0. \quad (3.17)$$

We claim that $\mathcal{L}_k^r(\mathfrak{a}_j) = \mathcal{L}_{s_0}^r(\mathfrak{a}_{s_0})$ for all $k \geq s_0$ and $j \geq s_0$. For $a \in \mathcal{L}_{s_0}^r(\mathfrak{a}_{s_0})$, pick $f \in \mathfrak{a}_{s_0} \subseteq \mathfrak{a}_j$ of degree s_0 with leading coefficient a , then $ft^{k-s_0} \in \mathfrak{a}_j$ has degree k with leading coefficient a , so $a \in \mathcal{L}_k^r(\mathfrak{a}_j)$.

For the converse inclusion, let $m := \max(j, k)$. Then

$$\mathcal{L}_k^r(\mathfrak{a}_j) \subseteq \mathcal{L}_k^r(\mathfrak{a}_m) \subseteq \mathcal{L}_m^r(\mathfrak{a}_m) = \mathcal{L}_{s_0}^r(\mathfrak{a}_{s_0}),$$

where the first inclusion is Lemma 3.15(2), the second uses the argument above applied to $\mathfrak{a}_m$, and the equality is (3.17).

For each $k \in \{0, \dots, s_0 - 1\}$, the chain

$$\mathcal{L}_k^r(\mathfrak{a}_1) \subseteq \mathcal{L}_k^r(\mathfrak{a}_2) \subseteq \dots$$

is an ascending chain of graded right ideals of R , so it stabilizes at some $\nu_k \geq 1$. Let $j_0 := \max(\nu_0, \dots, \nu_{s_0-1}, s_0)$. Then $\mathcal{L}_k^r(\mathfrak{a}_j) = \mathcal{L}_k^r(\mathfrak{a}_{j_0})$ for all $k \geq 0$ and $j \geq j_0$: for $k < s_0$ by stabilization, and for $k \geq s_0$ by the claim. Therefore, by Lemma 3.15(3), $\mathfrak{a}_j = \mathfrak{a}_{j_0}$ for all $j \geq j_0$. $\square$

Corollary 3.18. *If R is gr-left-Noetherian (resp. gr-right-Noetherian), then so is $R[t]$.*

Proof. Apply Proposition 3.16 with $\delta = 0$. $\square$

4 | Ring-Theoretic Invariance Under Graded Equivalence

Throughout this section, Γ denotes an *abelian* group. We establish that gr-simplicity and gr-primality are invariants of the homogeneous graded Morita equivalence relation $\approx_{\text{hgr}}$ of Definition 1.13, and that the differential polynomial structure of $R[t; \delta]$ is itself preserved under equivalence via a compatible idempotent. The abelian hypothesis is imposed so that we may apply the Homogeneously Graded Extended Morita Theorem of Abrams et al. [20]; as noted in the introduction, the algebraic core of that framework (the equivalence (C2) $\Leftrightarrow$ (C4)) does not require commutativity of Γ , and the full extension to arbitrary groups is expected once a proof of [20, Theorem 4.7] for non-abelian groups is available.

Proposition 4.1. *Let A and B be Γ -graded rings with $A \approx_{\text{hgr}} B$. Then A is gr-simple (resp. gr-prime) if and only if B is.*

Proof. By Definition 1.13, there exist $n \in \mathbb{N}$, a full homogeneous idempotent $\ell \in M_n(B)_e$, and a graded isomorphism $A \cong_{\text{gr}} \ell M_n(B) \ell$. Set $R := M_n(B)$.

Suppose A is gr-simple, and let $\mathfrak{b}$ be a nonzero proper graded ideal of B . Set $\mathfrak{d} := M_n(\mathfrak{b})$. Then $\mathfrak{d}$ is a nonzero proper graded ideal of R , and $\mathfrak{a} := \ell \mathfrak{d} \ell$ is a graded ideal of $\ell R \ell$. Since ℓ is full, $\mathfrak{a}$ is a nonzero graded ideal of $\ell R \ell \cong A$. By gr-simplicity of A , we have $\mathfrak{a} = \ell R \ell$. Then $\ell = \ell 1_R \ell \in \ell R \ell = \ell \mathfrak{d} \ell$. Since ℓ is full, $R = R \ell R \subseteq R(\ell \mathfrak{d} \ell)R \subseteq \mathfrak{d}$, so $R = \mathfrak{d}$, a contradiction. Therefore B is gr-simple. The converse follows by symmetry.

Suppose that A is gr-prime, and let $\mathfrak{b}_1, \mathfrak{b}_2$ be nonzero graded ideals of B such that $\mathfrak{b}_1 \mathfrak{b}_2 = 0$. For $i = 1, 2$, set $\mathfrak{a}_i := \ell M_n(\mathfrak{b}_i) \ell$. As above, each $\mathfrak{a}_i$ is a nonzero graded ideal of A . Moreover,

$$\mathfrak{a}_1 \mathfrak{a}_2 = \ell M_n(\mathfrak{b}_1) \ell \ell M_n(\mathfrak{b}_2) \ell \subseteq \ell M_n(\mathfrak{b}_1 \mathfrak{b}_2) \ell = 0,$$

which contradicts the gr-primality of A . Hence $\mathfrak{b}_1 = 0$ or $\mathfrak{b}_2 = 0$, and B is gr-prime. The converse again follows by symmetry. $\square$

The following proposition characterizes the graded rings homogeneously graded equivalent to $R[t; \delta]$ that arise from full idempotents in R_e , showing that they are again graded differential polynomial rings.

Proposition 4.2. Let T be a graded ring, δ a γ -derivation of T , and $\ell \in T_e$ an idempotent that is full in T_e and satisfies $\delta(\ell) = 0$. Define $S := \ell T \ell$ with induced grading $S_\tau := \ell T_\tau \ell$ for $\tau \in \Gamma_R$, and the map $\bar{\delta} : S \rightarrow S$ by $\bar{\delta}(s) := \ell \delta(s) \ell$ for $s \in S$. Then the following statements hold.

- (1) $\bar{\delta} \in \text{Der}_\gamma(S)$.
- (2) $\ell T[t; \delta] \ell \cong_{\text{gr}} S[t; \bar{\delta}]$ as graded rings.
- (3) $S[t; \bar{\delta}] \approx_{\text{hgr}} T[t; \delta]$.

Proof. (1) Let $s = \ell x \ell \in S_\tau$ for some $x \in T_\tau$. Since $\delta(\ell) = 0$ and $\ell^2 = \ell$, the Leibniz rule yields $\delta(\ell^2) = 0$. Thus,

$$\bar{\delta}(s) = \ell \delta(\ell x \ell) \ell = \ell (\delta(\ell) x \ell + \ell \delta(x) \ell + \ell x \delta(\ell)) \ell = \ell \delta(x) \ell.$$

As $\delta \in \text{Der}_\gamma(T)$, we have $\delta(x) \in T_{\tau\gamma}$, so $\ell \delta(x) \ell \in S_{\tau\gamma}$. Moreover, $\bar{\delta}$ satisfies the Leibniz rule

$$\bar{\delta}(s_1 s_2) = \ell \delta(s_1 s_2) \ell = \ell (\delta(s_1) s_2 + s_1 \delta(s_2)) \ell = (\ell \delta(s_1) \ell) (\ell s_2 \ell) + (\ell s_1 \ell) (\ell \delta(s_2) \ell) = \bar{\delta}(s_1) s_2 + s_1 \bar{\delta}(s_2),$$

since $s_i = \ell s_i = s_i \ell$ for $s_i \in S$. Hence $\bar{\delta} \in \text{Der}_\gamma(S)$.

(2) The Ore extension $T[t; \delta]$ is defined by the commutation relation $tx = xt + \delta(x)$ for $x \in T$, with grading $(T[t; \delta])_\tau = \bigoplus_{i \geq 0} T_{\tau\gamma-i} t^i$. Since $\delta(\ell) = 0$, we have $t\ell = \ell t$. By induction on i , using $\delta(\ell) = 0$ with (1.2), we then get

$$t^i \ell = \ell t^i, \quad i \geq 0. \quad (4.3)$$

Define $\Phi : S[t; \bar{\delta}] \rightarrow \ell T[t; \delta] \ell$ by

$$\Phi\left(\sum_{i=0}^n s_i t^i\right) := \sum_{i=0}^n s_i t^i,$$

where each $s_i \in S \subseteq T$ (viewed in $T[t; \delta]$).

For $s_i = \ell x_i \ell \in S$, we have $\ell s_i = s_i = s_i \ell$. Thus,

$$\ell \left(\sum_i s_i t^i \right) \ell = \sum_i (\ell s_i) (t^i \ell) = \sum_i s_i (\ell t^i) = \sum_i s_i t^i = \Phi \left(\sum_i s_i t^i \right),$$

by (4.3). Hence $\Phi(S[t; \bar{\delta}]) \subseteq \ell T[t; \delta] \ell$. The map Φ is clearly additive. For multiplication in $S[t; \bar{\delta}]$, $ts = st + \bar{\delta}(s)$. Then

$$\Phi(ts) = \Phi(st + \bar{\delta}(s)) = st + \bar{\delta}(s) = st + \ell \delta(s) \ell.$$

In $T[t; \delta]$, $ts = st + \delta(s)$, so $\Phi(ts) = ts - \delta(s) + \ell \delta(s) \ell = ts + (\ell \delta(s) \ell - \delta(s))$. But since $s = \ell x \ell$ for some $x \in T$ and $\delta(\ell) = 0$,

$$\delta(s) = \delta(\ell x \ell) = \ell \delta(x) \ell = \bar{\delta}(s),$$

so $\Phi(ts) = ts = t\Phi(s)$. Similarly, $\Phi(s_1 s_2) = \Phi(s_1)\Phi(s_2)$. Thus Φ is a graded ring homomorphism.

For surjectivity, let $f = \sum_i x_i t^i \in T[t; \delta]$ with $\ell f \ell \in \ell T[t; \delta] \ell$. Then

$$\ell f \ell = \sum_i (\ell x_i \ell) t^i = \sum_i (\ell x_i \ell) (\ell t^i) = \Phi \left(\sum_i (\ell x_i \ell) t^i \right),$$

by (4.3). So Φ is surjective. Injectivity follows directly.

(3) We show that ℓ is full in $(T[t; \delta])_e$. Since ℓ is full in T_e , there exist $a_j, b_j \in T_e$ with $\sum_j a_j \ell b_j = 1$. Let rt^k be an arbitrary pure element of $(T[t; \delta])_e$, with $r \in T_{\gamma-k}$. Then

$$rt^k = rt^k \left(\sum_j a_j \ell b_j \right) = \sum_j (rt^k a_j) \ell b_j.$$

By (1.2), $t^k a_j = \sum_{m=0}^k \binom{k}{m} \delta^m(a_j) t^{k-m}$ with $\delta^m(a_j) \in T_{\gamma m}$, so each summand $r \delta^m(a_j) t^{k-m}$ lies in $T_{\gamma m-k} t^{k-m} \subseteq (T[t; \delta])_e$; hence $rt^k a_j \in (T[t; \delta])_e$. Since $b_j \in T_e \subseteq (T[t; \delta])_e$, it follows that $(T[t; \delta])_e \ell (T[t; \delta])_e$ contains every pure element of $(T[t; \delta])_e$, so ℓ is full in $(T[t; \delta])_e$. By [20, Theorem 2.14], $M_\infty(T[t; \delta]) \cong_{\text{gr}} M_\infty(\ell T[t; \delta] \ell)$. Part (2) gives $\ell T[t; \delta] \ell \cong_{\text{gr}} S[t; \bar{\delta}]$, so $M_\infty(S[t; \bar{\delta}]) \cong_{\text{gr}} M_\infty(T[t; \delta])$. Therefore $S[t; \bar{\delta}] \approx_{\text{hgr}} T[t; \delta]$. $\square$

Remark 4.4. The condition $\delta(\ell) = 0$ in Proposition 4.2 is needed for the induced map $\bar{\delta}(s) := \ell \delta(s) \ell$ on $S := \ell T \ell$ to be a γ -derivation. Without this condition, the map $s \mapsto \ell \delta(s) \ell$ need not be a derivation of S , and consequently the differential polynomial ring $S[t; \bar{\delta}]$ need not be defined.

Nevertheless, this condition can always be achieved by an inner shift. Indeed, for an arbitrary $\delta \in \text{Der}_\gamma(T)$, set

$$r := \delta(\ell) \ell - \ell \delta(\ell) \in T_\gamma, \quad \delta' := \delta - \delta_r.$$

Since $\ell^2 = \ell$, we have $\delta(\ell) = \delta(\ell) \ell + \ell \delta(\ell)$ and $\ell \delta(\ell) \ell = 0$. It follows that $\delta_r(\ell) = \delta(\ell)$, hence $\delta'(\ell) = 0$.

Therefore Proposition 4.2 applies to δ' . If $\bar{\delta}'(s) := \ell \delta'(s) \ell$ for $s \in S$, then

$$\ell T[t; \delta] \ell \cong_{\text{gr}} \ell T[t; \delta'] \ell \cong_{\text{gr}} S[t; \bar{\delta}'].$$

Indeed, δ' differs from δ by the inner γ -derivation δ_r , so Corollary 2.8 gives $T[t; \delta] \cong_{\text{gr}} T[t; \delta']$; since this isomorphism fixes T , it restricts to the corresponding corners. Thus an arbitrary γ -derivation can always be replaced, up to graded isomorphism, by one satisfying the compatibility condition required in Proposition 4.2.

Proof of Theorem 1.14. For part (1): by Proposition 4.1, gr-simplicity and gr-primality are invariants of the homogeneous graded Morita equivalence class. Specifically, if $A \approx_{\text{hgr}} R[t; \delta]$ and $R[t; \delta]$ is gr-simple (resp. gr-prime), then A is gr-simple (resp. gr-prime).

For part (2): let $\ell \in R_e$ be a full idempotent with $\delta(\ell) = 0$. Proposition 4.2(2) gives $\ell R[t; \delta] \ell \cong_{\text{gr}} (\ell R \ell)[t; \bar{\delta}]$, and Proposition 4.2(3) gives $(\ell R \ell)[t; \bar{\delta}] \approx_{\text{hgr}} R[t; \delta]$. By (C2), every $A \approx_{\text{hgr}} R[t; \delta]$ arising from a full idempotent $\ell \in R_e$ is graded isomorphic to $\ell R[t; \delta] \ell$, hence to $(\ell R \ell)[t; \bar{\delta}]$. $\square$

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